Energy conservation and Poynting's theorem in the homopolar generator

Christopher F. Chyba[a)]
Department of Astrophysical Sciences and Woodrow Wilson School of Public and International Affairs, Princeton University, Princeton, New Jersey 08544

Kevin P. Hand[b)]
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91109

Paul J. Thomas[c)]
Department of Physics and Astronomy, University of Wisconsin, Eau Claire, Wisconsin 54702

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Most familiar applications of Poynting's theorem concern stationary circuits or circuit elements. Here, we apply Poynting's theorem to the homopolar generator, a conductor moving in a background magnetic field. We show that the electrical power produced by the homopolar generator equals the power lost from the deceleration of the rotating Faraday disk due to magnetic braking and review the way that magnetic braking arises within Poynting's theorem.
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I. INTRODUCTION

Poynting's theorem[1] for electric field $\mathbf{E}$, magnetic flux density $\mathbf{B}$, and current density $\mathbf{J}$ follows from Maxwell's equations (Faraday's and Ampère's laws) and the vector identity $\mathbf{E}\cdot(\nabla\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{B})$.[2,3] The theorem states that the rate at which work is done on the electrical charges within a volume is equal to the decrease in energy stored in the electric and magnetic fields, minus the energy that flowed out through the surface bounding the volume. In integral form, it can be written as

$$\int_V \mathbf{E}\cdot\mathbf{J}\,dV = -\frac{1}{2}\frac{\partial}{\partial t}\int_V\left(\epsilon E^2 + \frac{1}{\mu}B^2\right)dV - \frac{1}{\mu}\int_S(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}, \tag{1}$$

where S is the surface with area element $d\mathbf{a}$ bounding the volume V, and μ and ϵ are, respectively, the permeability and permittivity.

In Poynting's words, Eq. (1) means that "we must no longer consider a current as something conveying energy along the conductor. A current in a conductor is rather to be regarded as consisting essentially of a convergence of electric and magnetic energy from the medium upon the conductor and its transformation there into other forms."[1] This can be shown, for example, in the case of a wire segment of length L and radius r, with voltage drop ε carrying a steady current I. With $\mathbf{E} = (\varepsilon/L)\hat{\mathbf{z}}$ and $\mathbf{B} = (\mu I/2\pi r)\hat{\phi}$, integrating the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu$ over the cylindrical area $2\pi rL$ of the wire gives the power entering the wire's surface. The result is identical to that obtained from Ohm's law, with dissipated power $P = \varepsilon I$.[1,3] In addition to this example, Poynting's theorem has been applied to a number of stationary circuits of simple geometry.[4–8]

Can Poynting's theorem be applied to circuits in which a conductor is moving in the presence of a constant background magnetic field? The theorem should account for such cases, but elementary electromagnetism texts do not typically consider them. To examine a concrete and well-known system, we apply Poynting's theorem to the homopolar generator. We first demonstrate that the electrical power produced by the homopolar generator equals the power lost from the deceleration of the rotating Faraday disk due to magnetic braking. We then review how this magnetic braking arises within Poynting's theorem.

II. POWER DISSIPATION IN THE HOMOPOLAR GENERATOR

The homopolar generator, or Faraday disk, produces an electromotance (emf) by rotating a conducting disk in a constant uniform magnetic field.[9–11] The disk connects to an ammeter via brushes, one of which makes electrical contact with the rim of the disk and the other with the conducting axle of the disk. Consider the disk shown in Fig. 1, which has radius b and thickness h, with an axle of radius a aligned along the z-axis. The disk rotates with angular frequency ω in the laboratory frame, and the material of the disk has

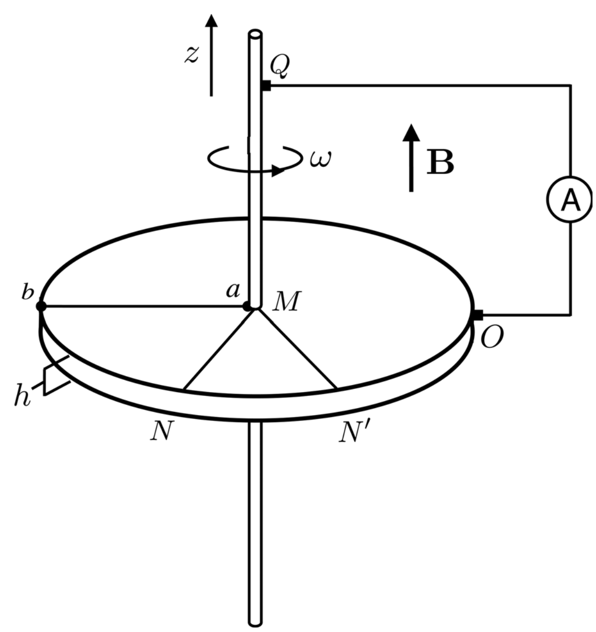

Fig. 1. The Faraday disk, or homopolar generator. An ammeter measures current flowing from point O to point Q.

conductivity σ. The disk rotates in a constant magnetic field of flux density $\mathbf{B} = B\hat{\mathbf{z}}$.

We first calculate the power dissipation in the disk, working in the lab frame. Divide the disk into concentric cylindrical shells of radial thickness dr and circumferential area $A = 2\pi rh$. The Lorentz force generates an emf across a shell given by

$$d\varepsilon = (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = \omega r B\, dr, \tag{2}$$

where we have used $\mathbf{v} = \omega r\hat{\phi}$. The emf across the entire disk is then

$$\varepsilon = \int_a^b d\varepsilon = \frac{1}{2}\omega B(b^2 - a^2). \tag{3}$$

The emf drives a current that runs from axle to rim, then through the ammeter back to the axle. Since only the disk is rotating in the lab frame, the only emf generated in the system is within the disk. The resistance of a single cylindrical shell is

$$dR = (\sigma A)^{-1} dr, \tag{4}$$

so the resistance of the entire disk is then

$$R = \frac{1}{\sigma}\int_a^b \frac{dr}{2\pi rh} = \frac{1}{2\pi\sigma h}\ln(b/a). \tag{5}$$

The Faraday disk is known as an intrinsically high-current low-voltage device,[12] and Eq. (5) can help us understand why. An application of Ohm's law gives $I/\varepsilon = R^{-1} = 2\pi\sigma h/\ln(b/a)$. Consider a disk made of a typical conductor, say aluminum, for which $\sigma = 3.8 \times 10^7$ S/m.[13] If I were 1 A, then even if $b = 10^4 a$, we could only have $\varepsilon = 1$ volt if h were 4×10^{-8} m. Because ε in Eq. (3) is fixed for a given $\mathbf{B}$, a, and b, the resistance of the remainder of the circuit only decreases I further. High voltages can be achieved by homopolar generators in astrophysical contexts where the length scales can be $\sim 10^3$ km or more. Within our Solar System, however, astrophysical homopolar generators appear to provide only minor electrical heating of planetary satellites.[14–18]

The total power P dissipated in the disk can be determined by integrating over the increments of power dP dissipated in each concentric shell. Using Eqs. (2) and (4), we have

$$dP = -\frac{(d\varepsilon)^2}{dR} = -2\pi\sigma B^2\omega^2 h r^3 dr, \tag{6}$$

where the negative sign indicates that power is being lost. Then

$$P = \int_a^b dP = -\frac{\pi}{2}\sigma B^2\omega^2 h(b^4 - a^4). \tag{7}$$

III. MAGNETIC BRAKING AND ENERGY CONSERVATION

The current density $\mathbf{J}$ in the disk interacts with $\mathbf{B}$ to decelerate the disk by magnetic braking. The magnetic braking force per unit volume is $\mathbf{F} = \mathbf{J} \times \mathbf{B}$.[12,19] The disk obeys Ohm's law for moving conductors

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{8}$$

so the braking force becomes

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} = \sigma[\mathbf{E} \times \mathbf{B} + (\mathbf{v} \times \mathbf{B}) \times \mathbf{B}]. \tag{9}$$

For $\mathbf{v} = \omega r\hat{\phi} = \omega r(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})$, $\mathbf{B} = B\hat{\mathbf{z}}$, and with no external electric field $\mathbf{E}$, we find

$$\mathbf{F} = -\sigma\omega r B^2\hat{\phi}. \tag{10}$$

The work done per unit volume is then $dW = \mathbf{F} \cdot d\mathbf{l} = \mathbf{F} \cdot \mathbf{v}\, dt$, so the rate at which work is done decelerating the disk is

$$\int_a^b \int_0^{2\pi} \int_0^h \mathbf{F} \cdot \mathbf{v}\, r\, dr\, d\phi\, dz = -\frac{\pi}{2}\sigma B^2\omega^2 h(b^4 - a^4), \tag{11}$$

which is identical to Eq. (7). Equation (11) makes clear that the power dissipated by the current driven to flow in the homopolar disk comes directly from the disk's kinetic energy of rotation. The magnetic field acts, so to speak, to convey the necessary energy from the rotation of the disk to the electrical circuit, but the magnetic field energy is unchanged.

In Eqs. (9) and (10), we treated $\mathbf{B}$ as the constant external field, ignoring secondary magnetic fields that must result from the generation of the current $\mathbf{J}$. Lorrain *et al.*[19] have examined the secondary magnetic fields $\mathbf{B}_{\text{disk}}$ and $\mathbf{B}_{\text{axle}}$ that arise as a result of the current flowing in the disk and axle, respectively. Both are azimuthal, so that $\mathbf{v} \times \mathbf{B}_{\text{disk}} = \mathbf{v} \times \mathbf{B}_{\text{axle}} = 0$. We note that neither component contributes to magnetic braking: $\mathbf{J} \times \mathbf{B}_{\text{disk}}$ is in the axial direction so it does not slow the rotating disk; and $\mathbf{J} \times \mathbf{B}_{\text{axle}}$ is in the radial direction so it does not slow the rotating axle.

IV. POYNTING'S THEOREM AND MAGNETIC BRAKING

To apply Poynting's theorem to the homopolar generator, it remains to show that the magnetic braking force (per unit volume) $\mathbf{J} \times \mathbf{B}$ is a consequence of Poynting's theorem, in which case the theorem would indeed show that the power production derives from the electromagnetic fields via magnetic braking. Davidson[20] gives a straightforward demonstration. One simply uses Ohm's law [Eq. (8)] to rewrite the first term in Poynting's theorem [Eq. (1)] as

$$\int_V \mathbf{E} \cdot \mathbf{J}\, dV = \frac{1}{\sigma}\int_V J^2\, dV + \int_V (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{v}\, dV, \tag{12}$$

where we have used $-(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{J} = (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{v}$. In our case, $\mathbf{E} = 0$ so in Eq. (12) the Joule heating $\int_V (J^2/\sigma)\, dV$ in the disk is equal to the energy lost from the disk by magnetic braking.

Other authors obtain magnetic braking from the Poynting theorem via the Lorentz transformations of the electromagnetic fields.[21–23] Define two frames: K is the frame in which a conductor is moving at velocity $\mathbf{v}$, and K' is the frame moving at $\mathbf{v}$ along with the conductor. How does the quantity $\mathbf{E} \cdot \mathbf{J}$ transform between the two frames? For $v^2 \ll c^2$, where c is the speed of light, we have

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \tag{13}$$

and

$$\mathbf{J}' = \mathbf{J}, \tag{14}$$

so

$$\mathbf{E}' \cdot \mathbf{J}' = \mathbf{E} \cdot \mathbf{J} - \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}). \tag{15}$$

Equation (14) is more complicated if $\mathbf{J}$ has a component parallel to $\mathbf{B}$, but for the Faraday disk that is not the case. Of course, the frame of a rotating disk is an accelerating frame and Maxwell's equations must be modified as a result, but these modifications are of order $(v/c)^2$ so we may ignore them here.[24] In K' Ohm's law is just $\mathbf{J}' = \sigma \mathbf{E}'$, so $\mathbf{E}' \cdot \mathbf{J}' = J'^2/\sigma = J^2/\sigma$ and, because $\mathbf{E} = 0$ in our case, Eq. (15) gives $J^2/\sigma = -(\mathbf{J} \times \mathbf{B}) \cdot \mathbf{v}$. Integration over the relevant volume once again shows that the Joule heating equals the energy lost from the disk by magnetic braking.

V. LINEAR ANALOG

For completeness, we now show that the same approach yields consistent results for the linear analog to the homopolar generator. Consider an infinitely long (along the x-axis) conducting rectangular bar moving with velocity $\mathbf{v} = v\hat{\mathbf{x}}$ through a field $\mathbf{B}$ perpendicular to the bar and direction of motion (Fig. 2).[25] The bar has finite width l and height h in the y and z directions, respectively. If the bar were instead finite along the x-axis and moving along connected stationary rails, it would be a so-called rail gun.[26–29]

The emf across an increment dl of the bar in the y-direction is $d\varepsilon = vBdl$. We calculate power dissipated per volume $dV = hwdl$, where w is some specified distance in the x-direction. Equation (4) gives $dR = dl/\sigma wh$, so by Eq. (6), the power dissipated per volume is

$$-\frac{(d\varepsilon)^2}{dR\, dV} = -\sigma v^2 B^2. \tag{16}$$

This is to be compared to the dissipated power calculated from magnetic braking. Using Ohm's law and following Eq. (9), the corresponding magnetic braking force per volume is

$$\mathbf{F} = -\sigma v B^2 \hat{\mathbf{x}}, \tag{17}$$

so the work done per unit volume is

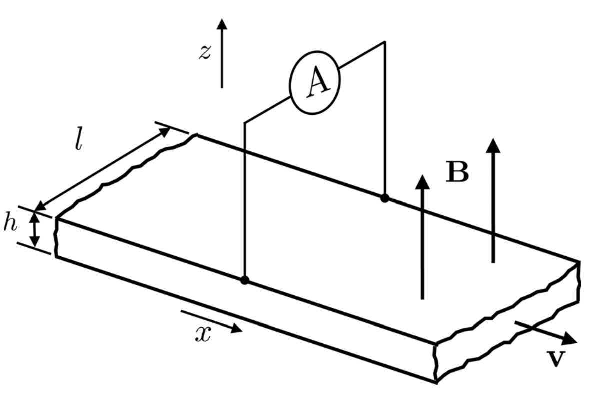

Fig. 2. The linear analog to the homopolar generator: A conducting bar moving with velocity $\mathbf{v}$ in the direction of its infinite length.

$$\mathbf{F} \cdot \mathbf{v} = -\sigma v^2 B^2, \tag{18}$$

which is identical to Eq. (16).

VI. CONCLUSION

Poynting's theorem has been successfully applied to the case of the homopolar generator (as well as its linear analog), in which a current flows in a circuit due to the motion of a conductor through a uniform background magnetic field. The power generated derives from the magnetic braking deceleration of the rotating Faraday disk. This magnetic braking is a natural consequence of Poynting's theorem.

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[a]Electronic mail: cchyba@princeton.edu
[b]Electronic mail: Kevin.P.Hand@jpl.nasa.gov
[c]Electronic mail: thomaspj@uwec.edu

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The "Dull" Power Shovel

The Physical Science Study Committee program was developed in the 1950s partly in reaction to textbooks that had become overly concerned with technology. One of the classic examples of this type of book was the 1955 edition of "Modern Physics" by Dull, Metcalfe and Brooks. The book contained what has now become a famous set of plastic overlays (the "Trans-Vision" process) detailing the operation of a power shovel, and this aspect of the book drew particular scorn. Now that this book and the reaction to it have been largely forgotten, it is interesting to look at the famous illustration once more. In the five views that are available by turning the overlays and looking on the right and left hand pages, the authors do a good job of illustrating applications of the simple machines: the three classes of levers, the pulley, the wheel and axle, the wedge, the screw and the inclined plane. (Notes by Thomas B. Greenslade, Jr., Kenyon College)