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Time-dependent embeddings for Schwarzschild-like solutions to the gravitational field equations

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An explicit formula for embedding the Schwarzschild solution in a three-dimensional flat space with indefinite metric for arbitrary Kruskal timelike coordinate v is presented. The time development of the Schwarzschild solution can then be represented by a succession of spacelike surfaces, each corresponding to a different value of v. It is seen that the standard representation of the Schwarzschild metric, the Flamm paraboloid, is in fact the $v = 0$ special case of a similar time-dependent embedding in a three-dimensional Euclidean space with positive definite metric. However, this embedding is inadequate in that it is not defined for most values of v. Thus, the embedding in a space with indefinite metric is to be preferred. The results for the Schwarzschild case are found to be readily extended to all metrics of a certain class, and a general embedding formula for arbitrary v results. Embeddings for the Schwarzschild, de Sitter, and Reissner–Nordström metrics are then special cases of this general form. It is seen that all such solutions behave similarly as v gets large. This suggests an alternate interpretation of the oscillatory character of the Reissner–Nordström "wormhole."

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I. INTRODUCTION

The Schwarzschild line element for a body of mass m (Ref. 1) is given by

$$ds^2 = -\Phi\, dt^2 + \Phi^{-1}\, dr^2 + r^2\, d\Omega^2, \tag{1}$$

where

$$\Phi = 1 - 2m/r \tag{2}$$

and

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{3}$$

is the metric of a unit sphere. Various methods have been employed to visualize the geometry of spacetime which arises from this solution. One aproach has been to embed the entire four-dimensional manifold in a flat space of higher dimension. Kasner[2] has shown that, excluding the trivial pseudo-Euclidean case, no four-dimensional manifold satisfying $R_{\mu\nu} = 0$ can be embedded in a five-dimensional flat space. However, Kasner,[3] and later Fronsdal,[4] have embedded (1) in a six-dimensional space. The geometry of the 4-manifold can then be pictured by taking subspaces of the higher-dimensional flat space.

A simpler approach is that first used by Flamm,[5] which takes advantage of the spherical symmetry of the Schwarzschild solution. Taking a constant-time slice of the $\theta = \pi/2$ plane yields the two-dimensional line element

$$ds^2 = \Phi^{-1}\, dr^2 + r^2\, d\phi^2, \tag{4}$$

which is then embedded by equating it to the metric of a three-dimensional Euclidean space[6] (positive definite metric):

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2. \tag{5}$$

Solving for dz^2 gives

$$dz^2 = (\Phi^{-1} - 1)\, dr^2, \tag{6}$$

which upon integration yields the well-known two-sheeted Flamm paraboloid:

$$z(r) = [8m(r - 2m)]^{1/2}. \tag{7}$$

This equation corresponds to a surface with the topology of an Einstein–Rosen bridge,[7] or "wormhole," connecting two asymptotically flat universes. The "throat" of the bridge has a narrowest region in the $z = 0$ plane, where the two universes join along a circle of circumference $4\pi m$, or, taking into account the θ-coordinate, along a sphere of surface area $16\pi m^2$.

The Reissner–Nordström solution for a body of mass m and electric charge q is given by an expression similar to (1):

$$ds^2 = -\Phi\, dt^2 + \Phi^{-1}\, dr^2 + r^2\, d\Omega^2, \tag{8}$$

where

$$\Phi = 1 - 2m/r + q^2/r^2 \tag{9}$$

and $d\Omega^2$ is as before. An identical procedure to that outlined above, with $m > |q|$, gives the embedding formula[8]:

$$z(r) = \int \left[\frac{1-\Phi}{\Phi}\right]^{1/2} dr = \int \left[\frac{2mr - q^2}{(r - r_+)(r - r_-)}\right]^{1/2} dr, \tag{10}$$

where $r_\pm = m \pm (m^2 - q^2)^{1/2}$.

Both these embeddings suffer from an inability to provide any geometrodynamic information, that is, neither can indicate how the curved space develops in time. Yet both the Schwarzschild and Reissner–Nordström solutions are known to exhibit quite dramatic time evolution. Kruskal diagrams[9] indicate that the Schwarzschild "throat" pinches off in a finite time[10] and the Reissner–Nordström "throat" oscillates between a minimum and maximum circumference of $2\pi r_-$ and $2\pi r_+$.[8]

In this paper, we develop a method for embedding any solution of the form

$$ds^2 = -\Phi\, dt^2 + \Phi^{-1}\, dr^2 + r^2\, d\Omega^2, \quad \Phi = \Phi(r) \tag{11}$$

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at an arbitrary, but explicit, Kruskal-like time coordinate v. That is, we are able to portray precisely, rather than merely qualitatively, embeddings which include the effectively *time-dependent* nature of certain black-hole type solutions. The time development of the solution can then be represented as a succession of spacelike surfaces, each surface corresponding to a different value of v. These surfaces are only defined for all v if the flat embedding space is endowed with an indefinite metric. It will be seen that the standard Schwarzschild and Reissner–Nordström embeddings discussed above are actually special cases, at time $v = 0$, of the embeddings which result from a similar procedure in which a flat space with positive definite metric is used. Such an embedding is found to be undefined (becomes imaginary) for most values of v. We suggest it is physically more appropriate, in representing solutions to the field equations, to use embeddings that avoid such behavior.

In Sec. II, we present two methods for obtaining such an embedding for the Schwarzschild metric (1). A succession of surfaces at different v is given, and the $v = 0$ surface is compared to the standard Flamm embedding. In Sec. III, with a slight extension of the general Kruskal-like transformations of Graves and Brill,[8] we generalize one of the methods of Sec. II to any metric of the form (11). In Sec. IV, we consider several special cases of this general form, including the Schwarzschild and Reissner–Nordström metrics. It is seen that all solutions of the form (11) must exhibit similar behavior as v goes to $\pm\infty$. Consideration of the dissimilar time evolutions of the Schwarzschild and Reissner–Nordström solutions, in the light of this result, suggests an alternate view of the oscillatory behavior of the Reissner–Nordström "wormhole." Rather than crediting the pulsation in time to the separate and opposing actions of gravitational pull and Maxwell pressure,[8] it is simpler to take the view that the portrayal of the full manifold which results from solving the equations $R_{\mu\nu} = -8\pi T_{\mu\nu}$ for a spherical mass endowed with charge requires a timelike coordinate v that is itself oscillatory.

II. EMBEDDING THE SCHWARZSCHILD METRIC AT ARBITRARY *V*

The well-known Kruskal transformation[9] giving the maximal analytic extension of the Schwarzschild solution is

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = (1 - r/2m)^{1/2} \exp(r/4m) \begin{Bmatrix} \sinh(t/4m) \\ \cosh(t/4m) \end{Bmatrix} \tag{12}$$

for $r < 2m$, and

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [(r/2m) - 1]^{1/2} \exp(r/4m) \begin{Bmatrix} \cosh(t/4m) \\ \sinh(t/4m) \end{Bmatrix} \tag{13}$$

for $r > 2m$. The line element, now free of the coordinate ("pseudo") singularity at $r = 2m$, becomes

$$ds^2 = f^2(-dv^2 + du^2) + r^2\, d\Omega^2, \tag{14}$$

where

$$f^2 = (32m^3/r) \exp(-r/2m), \tag{15}$$

and $d\Omega^2$ is as before.

Our goal is to embed the $\theta = \pi/2$ plane of (1),

$$ds^2 = -\Phi\, dt^2 + \Phi^{-1}\, dr^2 + r^2 d\phi^2, \tag{16}$$

into a flat space with metric given by

$$ds^2 = -dr^2 + dz^2 + r^2\, d\phi^2. \tag{17}$$

The choice of this particular metric will be discussed shortly. We eliminate dt^2 from (16) in such a way that the time dependence of the metric remains explicit. This is done by solving Eqs. (12) and (13) for t as a function of $v =$ const, differentiating, and squaring the result.[11] We obtain

$$dt^2 = \frac{v^2(r/2m)\, dr^2}{[v^2 - (1 - r/2m)\exp(r/2m)]\ [1 - r/2m]^2} \tag{18}$$

for r both greater and less than $2m$. Equation (16) then becomes

$$ds^2 = \left[\frac{(r/2m)\exp(r/2m)}{v^2 - (1 - r/2m)\exp(r/2m)}\right] dr^2 + r^2\, d\phi^2. \tag{19}$$

Equating Eqs. (19) and (17) gives the embedding formula:

$$dz = \left[\frac{(r/2m)\exp(r/2m)}{v^2 - (1 - r/2m)\exp(r/2m)} + 1\right]^{1/2} dr. \tag{20}$$

The same equation, with a useful intermediate result, is more easily obtained by setting $v =$ const. in Eq. (14). Equations (12) and (13) give

$$u^2 - v^2 = -(1 - r/2m)\exp(r/2m) \tag{21}$$

or

$$u = [v^2 - (1 - r/2m)\exp(r/2m)]^{1/2} \tag{22}$$

for all r. We therefore have the requirement that

$$v^2 \exp(-r/2m) \geqslant (1 - r/2m). \tag{23}$$

This inequality, which is independent of the signature of the space in which we embed our metric, is a particulary compact representation of the time evolution of the Schwarzschild solution, as shown in Fig. 1.

Differentiating (22) with $v =$ const, we obtain

$$du = (r/8m^2)\exp(r/2m)[v^2 - (1 - r/2m) \times \exp(r/2m)]^{-1/2}\, dr. \tag{24}$$

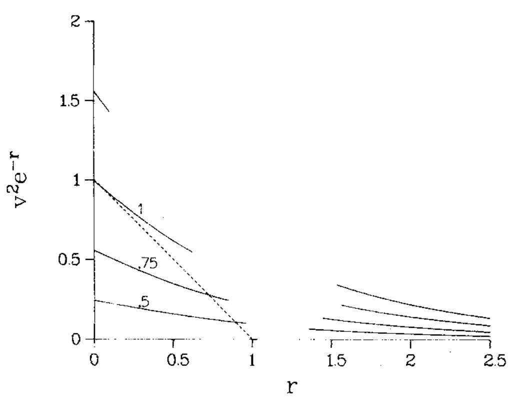

FIG. 1. The inequality $v^2e^{-r} \geqslant 1 - r$ (we have set $2m = 1$), a necessary condition for the Schwarzschild solution in Kruskal coordinates to be embedded for arbitrary v, is a particularly simple representation of the solution's development in time. The embedding is defined only when v^2e^{-r} (solid lines) is greater than $1 - r$ (dashed line). The number attached to each curve indicates the corresponding value of $|v|$. At $|v| = 0$, the "throat" has minimum radius 1; as $|v|$ increases, increasingly smaller values of r are allowed (the "throat" contracts). Finally, at $|v| = \pm 1$, r can equal zero (the "throat" pinches off).

Substituting this expression into (14) and equating to (17) yields the embedding formula (20) immediately.

Had we used the positive definite metric (5) for our flat embedding space, rather than the indefinite metric (17), we would have obtained

$$dz = \left[\frac{(r/2m)\exp(r/2m)}{v^2 - (1 - r/2m)\exp(r/2m)} - 1\right]^{1/2} dr \qquad (25)$$

as our embedding formula. At the Kruskal time $v = 0$, this reduces to

$$dz = \left[\frac{(r/2m)}{(r/2m) - 1} - 1\right]^{1/2} dr = (\Phi^{-1} - 1)^{1/2}\, dr, \qquad (26)$$

which is just the Flamm embedding (7). We therefore see that the Flamm paraboloid is a special case of the time-dependent embedding (25). However, it is clear that the square root in this equation becomes imaginary for many realizable values of r and v. First write Eq. (25) in the form

$$dz = \left[\frac{\exp(r/2m) - v^2}{v^2 - (1 - r/2m)\exp(r/2m)}\right]^{1/2} dr\,. \qquad (27)$$

Equation (23) guarantees that the denominator of this expression is positive. Equation (27) will therefore be undefined (have imaginary square root) whenever

$$v^2 > \exp(r/2m)\,. \qquad (28)$$

Such a result is unsatisfactory; we expect a physically acceptable representation of our curved space to be well defined for all-time v. This suggests that (20) is a more appropriate choice than (25), which in turn indicates that the $v = 0$ special case of (20) is a more appropriate embedding than the Flamm paraboloid.

This result is not surprising. We should expect the Schwarzschild line element to require a space of indefinite metric to be embedded for all v. In order to embed an n-dimensional surface given by

$$ds^2 = \sum_{u,v=0}^{n-1} g_{\mu\nu}\, dx_\mu\, dx_\nu \qquad (29)$$

in an m-dimensional flat space of arbitrary signature, with metric

$$ds^2 = \sum_{i=0}^{m-1} \alpha_i\, df_i^2\,, \qquad (30)$$

where $f_i = f_i(x)$ and $\alpha_i = \pm 1$, we must have

$$ds^2 = \sum_{u,v=0}^{n-1} g_{\mu\nu}\, dx_\mu\, dx_\nu = \sum_{i=0}^{m-1} \alpha_i\, df_i^2 = \sum_{i=0}^{m-1} \sum_{u,v=0}^{n-1} \alpha_i \frac{\partial f_i}{\partial x_\mu}\frac{\partial f_i}{\partial x_\nu} dx_\mu\, dx_\nu\,, \qquad (31)$$

whence,

$$g_{\mu\nu} = \sum_{i=0}^{m-1} \alpha_i \frac{\partial f_i}{\partial x_\mu}\frac{\partial f_i}{\partial x_\nu}\,. \qquad (32)$$

Symmetry of the metric tensor $g_{\mu\nu}$ in this equation gives $\frac{1}{2}n(n+1)$ first-order partial differential equations in the m unknowns $f_i(x)$. If there are no inconsistencies in the equations, we have the standard result that any n-dimensional manifold can always be embedded in a flat space of dimension $m \geqslant \frac{1}{2}n(n+1)$.[12] In the case of the Schwarzschild metric, we have $g_{00} = -1/g_{11}$ and the equations are not consistent. Equation (32) yields

$$g_{00} = -\Phi = \sum_{i=0}^{m-1} \alpha_i \frac{\partial f_i}{\partial x_0}\frac{\partial f_i}{\partial x_0}, \qquad (33)$$

and

$$g_{11} = \Phi^{-1} = \sum_{i=0}^{m-1} \alpha_i \frac{\partial f_i}{\partial x_1}\frac{\partial f_i}{\partial x_1}, \qquad (34)$$

which give

$$\sum_{i=0}^{m-1} \alpha_i \left(\frac{\partial f_i}{\partial x_0}\right)^2 = -\left[\sum_{i=0}^{m-1} \alpha_i \left(\frac{\partial f_i}{\partial x_1}\right)^2\right]^{-1}. \qquad (35)$$

However, given a positive definite metric ($\alpha_i = 1$ for all i),

$$\sum_{i=0}^{m-1} \alpha_i \left(\frac{\partial f_i}{\partial x_\nu}\right)^2 = \sum_{i=0}^{m-1} \left(\frac{\partial f_i}{\partial x_\nu}\right)^2 \geqslant 0 \qquad (36)$$

for any ν. Equation (35) therefore shows the impossibility of embedding the entire Schwarzschild manifold in a positive definite Euclidean space. The case $v = 0$ is, of course, an exception to this result. If $v = 0$, then $r > 2m$, and (13) shows that $t = 0$ identically for any allowable r. The Schwarzschild metric is then no longer indefinite (since $dt = 0$), and for this special case the entire manifold can thus be embedded.[13]

To show the time evolution of the Schwarzschild solution using embedding diagrams, we choose different constant values of v in (20). For a given v, the equation can then be integrated numerically to give a spacelike two-dimensional surface. It is clear from (20) that the time evolution of the manifold is symmetric in v about the value $v = 0$, and that, as v goes to $\pm\infty$, $z(r) = r$. Embeddings for illustrative values of v are shown in Figs. 2 and 3. Of particular interest is the $v = 0$ embedding, corresponding to the maximum size of the Schwarzschild "throat." At $v = 0$, (20) can be integrated exactly to give

$$z = \sqrt{2}\int \left[\frac{r-m}{r-2m}\right]^{1/2} dr = [2(r-m)(r-2m)]^{1/2} + \frac{\sqrt{2}}{2}\, m \log\left[\frac{(r-2m)^{1/2} + (r-m)^{1/2}}{(r-m)^{1/2} - (r-2m)^{1/2}}\right]. \qquad (37)$$

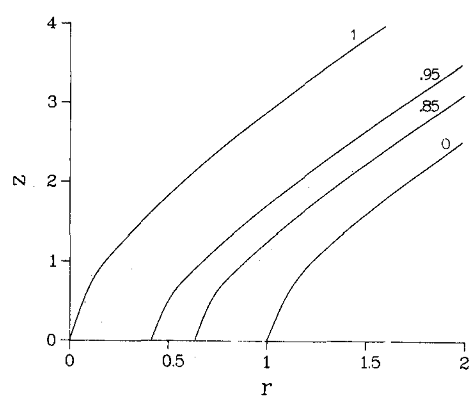

FIG. 2. Equation (20) gives an embedding of the Schwarzschild solution for any Kruskal-time v. Substituting into (20) a constant value of $|v|$, the equation can be numerically integrated to give $z = z(r)$. Here we show the embedding corresponding to $|v| = 0$ (maximum size of "throat"), $|v| = 0.85$, 0.95 ("throat" contracts), and $|v| = 1$ ("throat" pinches off). To obtain the entire two-sheeted embeddings, the curves must be rotated about the z axis, and reflected across the $z = 0$ plane. We have set $2m = 1$.

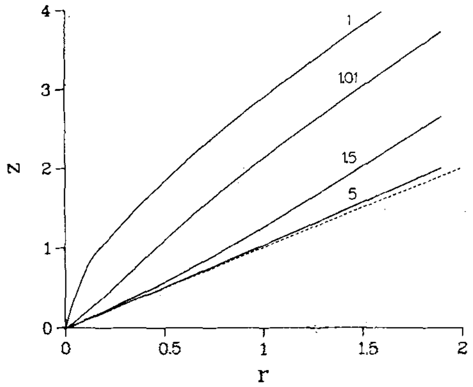

FIG. 3. Identical to Fig. 2, for the cases $|v| = 1, 1.01, 1.5, 5$. The Schwarzschild "throat" approaches the line $z = r$ as $|v|$ grows large.

This new $v = 0$ embedding is compared to the standard Flamm embedding in Fig. 4. It is seen that the behavior of the new embedding is qualitatively similar to that of Flamm: the "throat" has a narrowest radius of $r = 2m$ in the $z = 0$ plane, and the surface is asymptotically flat at large r.

III. THE GENERAL CASE

Graves and Brill[8] have given a general Kruskal-like transformation to remove pseudosingularities from metrics of the form (11), of which the Schwarzschild, de Sitter, and Reissner–Nordström metrics are special cases. It is assumed that $\phi(r)$ has zeroes or poles (the pseudosingularities) which are to be eliminated by transforming r and t to new coordinates $u(r,t)$ and $v(r,t)$, in terms of which light continues to travel along lines of slope ± 1. In such coordinates, the metric (11) takes the form

$$ds^2 = f^2(u,v)(du^2 - dv^2) + r^2(u,v)\, d\Omega^2, \tag{38}$$

where

$$f^2(u,v) = \Phi(r) \exp(-2\gamma r^*)/4A^2\gamma^2 \tag{39}$$

and

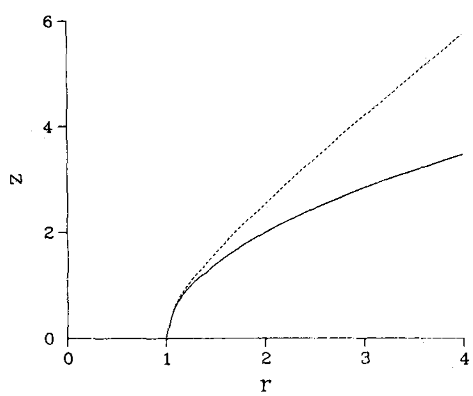

FIG. 4. The well-known Flamm paraboloid (7) is the $|v| = 0$ special case of an arbitrary v embedding into a space with positive definite metric (25), and is given by the solid line. The $|v| = 0$ special case of an embedding in a space of indefinite metric, (37), behaves similarly (dashed line); its minimum radius is 1, and it is asymptotically flat for large r. Both curves are to be rotated about the z axis and reflected through the $z = 0$ plane to give the full two-dimensional embedding.

$$r^* = \int dr/\Phi(r). \tag{40}$$

A is an arbitrary scale factor and γ is a constant chosen so that (39) is regular at the pseudosingularity (if more than one such singularity exists, several coordinate patches may be required). The coordinate transformation itself is given as

$$\begin{Bmatrix} u(r,t) \\ v(r,t) \end{Bmatrix} = 2A \exp(\gamma r^*) \begin{Bmatrix} \cosh(\gamma t) \\ \sinh(\gamma t) \end{Bmatrix} \tag{41}$$

with the inverse transformation given implicitly by

$$u^2 - v^2 = 4A^2 \exp(2\gamma r^*), \tag{42}$$

$$t = (1/2\gamma)\tanh^{-1}[2uv/(u^2 - v^2)]. \tag{43}$$

Equation (42) gives

$$u = [v^2 + 4A^2 \exp(2\gamma r^*)]^{1/2}. \tag{44}$$

Differentiating this equation and substituting into (38), with $v = \text{const.}$, yields

$$ds^2 = 4A^2 \Phi^{-1}(r) \exp(2\gamma r^*) \times [v^2 + 4A^2 \exp(2\gamma r^*)]^{-1} dr^2 + r^2\, d\Omega^2. \tag{45}$$

Equating (45) and (17) then gives

$$dz = \left[\frac{4A^2 \exp(2\gamma r^*)}{\Phi(r)[v^2 + 4A^2 \exp(2\gamma r^*)]} + 1 \right]^{1/2} dr. \tag{46}$$

We therefore have a general procedure for embedding any metric of the form (11) at arbitrary time v. Finally, we note that (44) provides the general requirement

$$v^2 \geqslant -4A^2 \exp(2\gamma r^*). \tag{47}$$

IV. APPLICATIONS

For the Schwarzschild metric, Graves and Brill put

$$\gamma = 1/4m, \quad A = \tfrac{1}{2}, \quad \Phi = (1 - 2m/r),$$
$$r^* = r + 2m \log(r - 2m). \tag{48}$$

These values give the transformation equations

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = (r - 2m)^{1/2} \exp(r/4m) \begin{Bmatrix} \cosh(t/4m) \\ \sinh(t/4m) \end{Bmatrix}. \tag{49}$$

Clearly, however, these equations are not valid when $r < 2m$. We therefore choose $r^* = r + 2m \log|r - 2m|$ in general, and, in addition to (49), take

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = 2A \exp(\gamma r^*) \begin{Bmatrix} \sinh(\gamma t) \\ \cosh(\gamma t) \end{Bmatrix} \tag{50}$$

for the Schwarzschild metric in the case $r < 2m$. The inverse transformation (42)—and hence our embedding formula—now remains unique regardless of the value of r. Finally, to bring our results completely in line with the transformation of Kruskal, we take $A = 1/(8m)^{1/2}$. Substitution of these values into (42) reveals that the Schwarzschild embedding of Sec. II is a special case of the general procedure presented in Sec. III.

As a second example, consider the metric of the de Sitter universe in the static frame.[14] We have

$$\Phi = 1 - r^2/R^2, \tag{51}$$

where $0 < r < R$. We restrict our discussion of this metric to the inequality (47), which, with

$$r^* = \frac{R}{2}\log\left(\frac{R+r}{R-r}\right); \quad \gamma = -\frac{1}{R}; A = 1\,, \tag{52}$$

becomes

$$v^2(R+r) \geqslant 4(r-R)\,. \tag{53}$$

This inequality indicates that r cannot become infinite unless $|v| \geqslant 2$, in agreement with the usual result.[15]

Finally, we consider the Reissner–Nordström metric (8), restricting ourselves to the case in which the mass exceeds the value associated by general relativity with the charge

$$m > |q|\,, \tag{54}$$

where both are in units of centimeters. While such a restriction avoids so-called "naked" singularities,[16] the physical significance of this metric remains unclear. Misner and Wheeler[17] have shown the condition (54) to be incompatible with a nonclassical description of charge and mass. In addition, it has recently been shown that a gravitational collapse to the Reissner–Nordström singularity is impossible for a broad class of boundary-surface histories.[18]

With the condition (54), the metric has two pseudosingularities at

$$r_\pm = m \pm (m^2 - q^2)^{1/2}. \tag{55}$$

Two coordinate patches (i, j) are thus required in the neighborhoods of r_+ and r_-. Graves and Brill give

$$r^* = r + \left(\frac{r_+^2}{r_+ - r_-}\right)\log(r - r_+) - \left(\frac{r_-^2}{r_+ - r_-}\right)\log(r - r_-) \tag{56}$$

and

$$\gamma_i = (r_i - r_j)/2r_i^2\,, \tag{57}$$

which yield the transformation

$$\begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = 2A\,(r - r_i)^{1/2}(r - r_j)^{\alpha_j}\exp(\gamma_i\, r)\begin{Bmatrix} \cosh \gamma_i\, t \\ \sinh \gamma_i\, t \end{Bmatrix}, \tag{58}$$

where

$$\alpha_j = -\tfrac{1}{2}(r_j/r_i)^2\,, \tag{59}$$

with $(i, j) = (+, -)$ or $(-, +)$. As in the Schwarzschild case, however, these equations need to be generalized for values of r other than $r > r_+ > r_-$. Our criterion is that the inverse transformation (42) remains unique for each coordinate patch. Thus, our transformations becomes

$$r^* = r + \left(\frac{r_+^2}{r_+ - r_-}\right)\log|r - r_+| - \left(\frac{r_-^2}{r_+ - r_-}\right)\log|r - r_-| \tag{60}$$

and

$$\begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = 2A\,|r - r_i|^{1/2}|r - r_j|^{\alpha_j}\exp(\gamma_i\, r)\beta_i\,, \tag{61}$$

where α_j is as before and

$$\beta_i = \begin{Bmatrix} \cosh \gamma_i\, t \\ \sinh \gamma_i\, t \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \sinh \gamma_i\, t \\ \cosh \gamma_i\, t \end{Bmatrix}, \tag{62}$$

depending on the sign of $|r - r_i||r - r_j|^{2\alpha_j}$ relative to $(r - r_i)(r - r_j)^{2\alpha_j}$.

We can substitute these values into (46) to obtain an embedding for the Reissner–Nordström solution at arbitrary v. In particular, the $v = 0$ embedding

$$z(r) = \int \left[\frac{1 + \Phi(r)}{\Phi(r)}\right]^{1/2} dr \tag{63}$$

differs from the Flamm-like embedding (10), which results from a flat space with positive definite metric.

Rather than utilizing two coordinate patches, we could, at least formally, follow a similar embedding procedure in the extended Reissner–Nordström manifold. Here the metric (8) may be written[19] in the form

$$ds^2 = F^2(-d\psi^2 + d\xi^2) + r^2\, d\Omega^2\,, \tag{64}$$

where

$$F = F(\psi, \xi); \quad r = r(\psi, \xi) \tag{65}$$

and $d\Omega$ is the usual spherical surface element. However, the complicated nature of the transformations (65) indicates that it is in practice simpler to use the series of coordinate patches (i, j) given by Graves and Brill.

Finally, we wish to consider the time development of the manifold. Consider the general embedding equation (46), of which that of the Reissner–Nordström metric is a special case. It is clear from (46) that, as we let the absolute value of v in our constant time embeddings grow large, the equation goes to

$$z(r) = \int dr = r\,, \tag{66}$$

which is pinched off at $r = 0$ in the $z = 0$ plane. In particular, this same behavior holds for the Reissner–Nordström embedding. Yet it is known that the radius of the "throat" for this metric must pulsate periodically in time. This pulsation has been credited to a "cushioning" by Maxwell pressure of the electric field through the "throat."[8] From a consideration of the embedding formula, however, in which the effect of the presence of electric charge is taken into account by the values assigned r^* and γ, it seems the "throat" must pinch off as in the Schwarzschild case. This does not take place because $|v|$ never goes to infinity; for an observer on the "throat" ($u = 0$ in the first patch), v reaches a maximum value of

$$v^2 = 4A^2 \exp(2\gamma_+ r_c)(r_+ - r_c)(r_c - r_-)^{2\alpha}\,, \tag{67}$$

where

$$\alpha_- = -\frac{1}{2}\left(\frac{r_-}{r_+}\right)^2, \quad \gamma_+ = \left(\frac{r_+ - r_-}{2r_+^2}\right), \tag{68}$$

and

$$r_+ > r_c > r_-\,. \tag{69}$$

At this value of $r = r_c$, the observer crosses into the second patch. Upon return to a patch identical to the first, the observer moves only between two finite values of v, again departing the patch at a time v given by (67). That is, $|v|$ never approaches infinity, but rather, oscillates between finite values. We adopt the view that the Reissner–Nordström "throat" pulsates because the timelike coordinate needed to describe both patches of the manifold which results from a spherically symmetric mass and charge distribution must itself be oscillatory.

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